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Four-Vector Representation of Fundamental Particles

Joachim Lambek[1]

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Modifying a proposal by Harari and Shupe, we associate with each lepton, quark, and gauge boson a *fundamental* 4-vector with entries ± 1 or 0. A Feynman vertex corresponds to the addition of two fundamental vectors giving rise to a third.

In 1979, Harari [5] and Shupe [9] independently proposed a theory according to which leptons and quarks are made up of two so-called *preons*, called *rishons* T and V by Harari and *quips* $+$ and 0 by Shupe, their antiparticles being denoted by $\overline{T}$ and $\overline{V}$ or by $-$ and $\overline{0}$, respectively. The idea was that every lepton is a triple of three identical preons or antipreons, while every quark is a triple of three nonidentical ones. Since there are exactly three arrangements, say, of two Ts and one V, there are three different up-quarks, said to have different colors. Similarly, one $\overline{T}$ and two $\overline{V}$s make up three differently colored down-quarks.

I suggest that the preons T and V be replaced by the integers 1 and 0, as is perhaps implicit in Shupe's notation, and that their alleged antiparticles be replaced by -1 and $-0 = 0$, respectively. The last equation is not envisaged by Shupe, who distinguishes between $\overline{0}$ and 0, and it would associate the same vector (0, 0, 0) with both neutrino and antineutrino. To cicumvent this problem, we will distinguish between fermions and their antiparticles with the help of a fourth integer, provisionally called the *fermion number*; it is 1 for leptons and quarks and -1 for their antiparticles. Both Harari and Shupe require 6-tuples of preons to describe the structure of gauge bosons; here we simply put their fermion number equal to 0.

[1] Mathematics Department, McGill University, Montreal, Quebec H3A 2K6, Canada; e-mail: lambek@triples.math.mcgill.ca

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Formally, we associate with each fundamental particle a *fundamental* 4-vector $a = (a_0, a_1, a_2, a_3)$, where each $a_i = 1, -1$, or 0. We may also write $a = a_0 + ia_1 + ja_2 + ka_3$, where $1 = (1, 0, 0, 0)$, $i = (0, 1, 0, 0)$, etc., are unit vectors. Here a_0 is the fermion number, and the charge of a particle may be calculated as

$$\frac{e}{3}(a_1 + a_2 + a_3)$$

where $-e$ is the charge of an electron.

What is presented here is not an analysis of the substructure of the fundamental particles, but what appears to be a useful bookkeeping device. Moreover, we shall confine attention to the first generation of particles; an explanation of the higher generations may be found in Adler's [1, 2] much more sophisticated theory.

The following table shows which 4-vector is associated with each fundamental particle. The letters u, d, and g stand for up-quarks, down-quarks, and gluons, respectively, and the subscripts R, B, and G refer to the colors red, blue, and green, respectively.

Fermions

e^-: $1 - i - j - k$

ν: 1

u_R: $1 + j + k$

u_B: $1 + i + k$

u_G: $1 + i + j$

d_R: $1 - i$

d_B: $1 - j$

d_G: $1 - k$

Antifermions

e^+: $-1 + i + j + k$

$\bar{\nu}$: -1

$\bar{u}_R$: $-1 - j - k$, etc.

Gauge bosons
γ, Z^0: 0
W^-: $-i - j - k$
W^+: $i + j + k$
g_{BG}: $j - k$
g_{GR}: $-i + k$
g_{RB}: $i - j$
g_{GB}: $-j + k$
g_{RG}: $i - k$
g_{BR}: $-i + j$

Here, for example, g_{BG} is the gluon responsible for changing color from blue to green. We have not accounted for the two remaining gluons, which do not effect the color of a quark.

We have thus associated 25 different 4-vectors to fundamental particles; but the photon and the Z^0 correspond to the same 4-vector 0. Altogether there are $3^4 = 81$ fundamental 4-vectors and there is room to accommodate some as yet undiscovered particles, such as those proposed in supersymmetry.

The fundamental 4-vectors live in the Abelian group $\mathbb{Z}^4$. Sometimes the sum of two fundamental vectors is again fundamental; we claim that such equations account for all possible Feynman vertices. Thus $a + b = c$ would correspond to the diagram

$$\begin{array}{cc} a & \\ \searrow & c \\ & \longrightarrow \\ \nearrow & \\ b & \end{array}$$

However, it is customary to replace the straight arrows by wavy arrows for gauge bosons. Since we are dealing with a group under addition, the following diagrams are equivalent to the above:

$$\begin{array}{cc} a & \\ \searrow & -c \\ & \longrightarrow, \\ \swarrow & \\ -b & \end{array} \qquad \begin{array}{cc} -a & \\ \nwarrow & c \\ & \longrightarrow \\ \nearrow & \\ b & \end{array}$$

For example, when $b = 0$ and $c = a$, the last two diagrams describe pair annihilation and pair creation, respectively.

We illustrate the usefulness of vector addition by looking at some double events, all taken from Gell-Mann [4].

(a) Two electrons exchanging a virtual photon, giving rise to the force between them:

$$1 = 1 + 0, \qquad 0 + (-1) = -1$$

(b) The force between quarks from a gluon exchange:

$$1 + i + j = (1 + j + k) + (i - k), \qquad (i - k) + (1 - i) = 1 - k$$

$$1 - i = (l - j) + (-i + j), \qquad (-i + j) + (1 - j) = 1 - i.$$

(c) An electron turning into a neutrino and a W^-, while an up-quark turns into a down-quark:

$$1 - i - j - k = 1 + (-i - j - k),$$

$$(-i - j - k) + (1 + j + k) = 1 - i$$

(d) The scattering of a neutrino off a down-quark mediated by a Z^0:

$$1 = 1 + 0, \qquad 0 + (1 - i) = 1 - i$$

For example, (b_1) and (c) are illustrated by the following Feynman diagrams, where the usual wavy arrows have been replaced by dashed ones:

(b_1)

$$\begin{array}{ccc} u_R & & d_G \\ \nwarrow & g_{RG} & \nearrow \\ & \dashrightarrow & \\ \nearrow & & \nwarrow \\ u_G & & d_R \end{array}$$

(c)

$$\begin{array}{ccc} \nu & & d_R \\ \nwarrow & W^- & \nearrow \\ & \dashrightarrow & \\ \nearrow & & \nwarrow \\ e^- & & u_R \end{array}$$

One is tempted to extend the 4-vector notation to composite particles, e.g., associating the vectors 3 and $3 - i - j - k$ with the neutron and proton, respectively, both with fermion number 3. Thus, the neutron decay

$$n \to p^+ + e^- + \bar{\nu}$$

would be illustrated by the equation

$$3 = (3 - i - j - k) + (1 + i + j + k) + (-1)$$

Historically, a consideration such as this led Pauli to propose the existence of neutrinos in the first place. This decay can also be explained at a more fundamental level if we decompose the neutron and the proton into quarks:

$$\begin{aligned} 3 &= (1 - i) + (1 + i + j) + (1 - j) \\ &= (-i - j - k) + (1 + j + k) + (1 + i + j) + (1 - j) \\ &= (1 - i - j - k) + (-1) + (3 + i + j + k) \end{aligned}$$

Unfortunately, the extended notation would fail to distinguish, e.g., between the weak gauge boson W^- of spin 1 and Yukawa's meson π^- of spin 0.

In general, the fermion number of a colorless composite is the lepton number plus three times the baryon number. Indeed, our bookkeeping procedure would suggest that these two numbers are not preserved separately. This would be verified, e.g., if a down-quark could be transformed into a neutrino with the help of an as-yet-undiscovered gauge boson, $(1 - i) + i = 1$, as was indeed speculated by Feynman [3].

The above discussion exploits the additive group of 4-vectors. However, our notation suggests that we are dealing with quaternions, which also have a multiplication such that

$$i^2 = j^2 = k^2 = ijk = -1$$

turning the group $\mathbb{R}^4$ into a division ring. In fact, Adler [2] asserts that his interest in the Harari–Shupe speculations led him to investigate quaternionic quantum mechanics, in which the wave function lives in a quaternionic Hilbert space. As I showed in my expository article [7], quaternions and biquaternions have been used successfully to describe the dynamics of electrons and photons for quite some time, and in ref. 8, I attempted to extend this approach to other fundamental particles.

In this article I have presented a modified form of the Harari–Shupe proposal. However, rather than claiming to represent the internal structure of quarks and leptons, I prefer to view this notation as a grammatical restriction on the language describing the interactions between elementary particles. It thus plays a role rather like dimensional analysis did for classical mechanics: this assigned to each physical quantity a 3-vector with integer components (or rational components if electromagnetic phenomena were included).

One final question remains: which of the 81 fundamental 4-vectors are associated with actual particles? It seems that all the known fundamental particles (and even the undiscovered gauge boson suggested by Feynman) are subject to the following empirical rule: The number of positive entries in the associated 4-vector is either zero or odd, and ditto for the number of negative entries. I would like to ascribe this rule to Benjamin Franklin, because its enunciation depends on his arbitrary decision of which form of electricity was to be called positive or negative, respectively. However, according to the theory known as supersymmetry, perhaps there is no restriction on the fundamental 4-vectors.

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